Limit theorem for randomly indexed sequence of random processes

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Abstract In this paper is proved the limit theorem for randomly indexed sequence of random processes in the case where sequences of random index and random processes are independent, also the estimation of convergence rate is obtained.

1 Introduction

The study of asymptotic behavior of randomly indexed sequence enjoy considerable attention in connection with applications in the theory of queues, Markov processes, word space modeling. In the papers [1-4] the conditions of convergence and asymptotic behavior of randomly indexed random variables is studied. In this work we consider the sequences of randomly indexed random processes, defined in Skorokhod space $D[0,1]$. The obtained limit theorem makes it easy to get the limit theorems for some random processes with random substitution defined in $D[0,1]$. The convergence conditions are quite weak, which allows to apply the theorem in many applications. The estimation of convergence rate is also obtained.

2 Main results

Recall the definition of the Skorokhod space and the metric in it. Denote by $\Delta[0,1]$ class of strictly increasing continuous mapping of the segment $[0,1]$ on it self such that $\lambda(0)=0$, $\lambda(1)=1$; $D[0,1]$ is a Skorokhod space, i.e. the space of functions defined in the segment $[0,1]$ and taking values in $\mathbb{R}$, right-continuous and with a finite limit on the left. In the space $D[0,1]$ we will consider the Skorokhod's metric

$$\rho(x,y) =$$

$$\inf\{\varepsilon > 0 : \exists\lambda \in \Delta[0,1],\ \sup_{0\le t\le 1} |x(t) - y(\lambda(t))| \le \varepsilon,\ \sup_{0\le t,s\le 1, s\ne t} \left| ln\frac{\lambda(t)-\lambda(s)}{t-s} \right| \le \varepsilon\},$$

$x, y \in D[0,1]$. It is well know [6] that this metric turns $D[0,1]$ into Polish space. Then (see, for example, [5]) cylindrical σ- algebra coincides with Borel σ- algebra, and the Borel σ- algebra [6] is generated by maps of type $f \to f(a)$ for $a \in [0,1]$.

Theorem 1 *Let Y_n be a sequence of random processes in $D[0,1]$ such that*

$$Y_n \xrightarrow{d} Y \text{ as } n \to \infty,$$

and trajectories of Y are continuous, $\nu_n(t)$ is a sequence of random processes in $D[0,1]$ with non-decreasing non-negative trajectories, for every $t \in [0,1]$ $\nu_n(t)$ take values in $\mathbb{N}$, $f(n)$ is a function taking values in $\mathbb{R}$ such that $f(n) \to \infty$ as $n \to \infty$ and

$$\frac{\nu_n}{f(n)} \xrightarrow{d} \nu \text{ in } D[0,1] \text{ as } n \to \infty,$$

and besides it exists $c > 0$ such that $\nu(t) > c$ for all $t \in [0,1]$. Let Y_n and ν_n are independent. Then

$$Y_{\nu_n} \xrightarrow{d} Y \text{ as } n \to \infty \text{ in } D[0,1], \tag{1}$$

where the convergence holds in the uniform norm.

To prove this theorem we need the following technical preliminary result.

Lemma 1 *Let $x_n(t)$ be a sequence of functions in $D[0,1]$ such that $x_n \to x$ in $D[0,1]$ as $n \to \infty$, x is continuous and the sequence of càdlàg non-decreasing non-negative functions $\mu_n(t)$ for every $t \in [0,1]$ take values in $\mathbb{N}$ and*

$$\frac{\mu_n}{f(n)} \xrightarrow{d} \mu \text{ in } D[0,1] \text{ as } n \to \infty$$

for some function $f(n)$, where $f(n) \to \infty$ as $n \to \infty$ and besides it exists $c > 0$ such that $\mu(t) > c$ for all $t \in [0,1]$. Then

$$x_{\mu_n} \to x \text{ in } D[0,1] \text{ as } n \to \infty,$$

where the convergence holds in the uniform norm.

Proof. Let $0 < \varepsilon < c$ is arbitrary. The convergence $x_n \to x$ implies the existence of n_1 such that for all $n > n_1$ $\rho_D(x_n, x) < \varepsilon$.

The convergence $\frac{\mu_n}{f(n)} \xrightarrow{d} \mu$ in $D[0,1]$ as $n \to \infty$ implies that exists n_2 such that for all $n > n_2$ it exists $\lambda_n \in \Delta[0,1]$ and

$$\sup_{0\le t\le 1}\left|\frac{\mu_n(t)}{f(n)} - \mu(\lambda_n(t))\right| < \varepsilon, \quad \sup_{0\le t,s\le 1, s\ne t}\left|ln\frac{\lambda(t)-\lambda(s)}{t-s}\right| < \varepsilon.$$

Then for all $t \in [0,1]$ and $n > n_2$ holds the inequality: $\mu_n(t) > (c-\varepsilon)f(n)$. Thus $f(n) \to \infty$ as $n \to \infty$, it exists n_3 such that for all $n > n_3$ we can assume $f(n) > \frac{\max\{n_1,n_2\}}{c-\varepsilon}$.

Then for all $n > n_3$

$$\rho_D(x_{\mu_n}, x) < \varepsilon,$$

which implies the lemma assertion.

Proof of the theorem. By Skorokhod theorem about one probability space (see, for example, theorem 11 in section V in [5]), it exists the probability space $(\Omega_1', \mathfrak{A}_1, P_1)$ and the random processes $X_n : \Omega_1' \to D[0,1]$ such that

$1) X_n \stackrel{d}{=} Y_n, \quad X \stackrel{d}{=} Y;$

$2) X_n \xrightarrow{a.s.} X$ at $n \to \infty$ in $D[0,1]$.

Denote by Ω_1 the measurable subset of Ω_1' such that $P_1(\Omega_1) = 1$ and the convergence 2) is true for all $\omega_1 \in \Omega_1$.

Also by Skorokhod theorem the convergence $\frac{\nu_n}{f(n)} \xrightarrow{d} \nu$ at $n \to \infty$ in $D[0,1]$ implies the existence of probability space $(\Omega_2', \mathfrak{A}_2, P_2)$ and random processes $\mu_n : \Omega_2' \to D[0,1]$ such that

$1) \nu_n \stackrel{d}{=} \mu_n, \quad \nu \stackrel{d}{=} \mu;$

$2) \frac{\mu_n}{f(n)} \xrightarrow{a.s.} \mu$ as $n \to \infty$ in $D[0,1]$.

Note that the set

$$A = \{f \in D[0,1] : f(t) \ge 0,\ f(t) \text{ is a non-decreasing function, taking values in } \mathbb{N}\}$$

is measurable relating Borel σ - algebra in D[0,1]. Then $1 = P(\nu_n \in A) = P(\mu_n \in A)$, that is almost all trajectories of μ_n are also non-negative, non-decreasing and take values in $\mathbb{N}$. Similarly we can see that almost sure $\mu(t) > c$ for all $t \in [0,1]$.

Denote by Ω_2 the measurable subset of Ω_2' such that $P_2(\Omega_2) = 1$, all trajectories of μ_n are non-negative and non-decreasing, the convergence 2) and inequality $\mu(t) > c$, $t \in [0,1]$ are true for all $\omega_2 \in \Omega_2$.

Further we will consider the probability space $(\Omega, \mathfrak{A}, P)$, where $\Omega = \Omega_1 \times \Omega_2$, the σ-algebra $\mathfrak{A}$ consists of elements of σ-algebra $\mathfrak{A}_1 \times \mathfrak{A}_2$ belonging to Ω, the probability P is a restriction of probability $P_1 \otimes P_2$ to σ-algebra $\mathfrak{A}$.

Let $\omega = (\omega_1, \omega_2) \in \Omega$. Consider the sequence $x_n \equiv X_n(\omega_1)$, $x \equiv X(\omega_1)$, $\gamma_n(t) \equiv \mu_n(t)(\omega_2)$, $\gamma(t) \equiv \mu(t)(\omega_2)$.

Then by Lemma 1 we obtain the convergence $x_{\gamma_n} \to x$ as $n \to \infty$ in $D[0,1]$ for all $\omega \in \Omega$, which implies the convergence $X_{\mu_n} \overset{a.s.}{\to} X$ as $n \to \infty$ in $D[0,1]$. Because the distributions of X_{μ_n} and Y_{ν_n} coincide, the assertion of theorem is proved.

Remark 1 *Note, that Theorem 1 not holds if trajectories of Y not continuous.*

Example 1 *Consider the sequence of non-random functions*

$$x_{2n}(t) = \begin{cases} 0 \ \textit{if} & 0 \le t \le \frac{1}{2} - \frac{1}{2^{2n}}, \\ 2^{2n}t + 1 - 2^{2n-1} \ \textit{if} & \frac{1}{2} - \frac{1}{2^{2n}} < t \le \frac{1}{2}, \\ 1 \ \textit{if} & \frac{1}{2} < t \le 1, \end{cases}$$

$$x_{2n+1}(t) = \begin{cases} 0 \ \textit{if} & 0 \le t \le \frac{1}{2}, \\ 2^{2n+1}t - 2^{2n} \ \textit{if} & \frac{1}{2} < t \le \frac{1}{2} + \frac{1}{2^{2n+1}}, \\ 1 \ \textit{if} & \frac{1}{2} + \frac{1}{2^{2n+1}} < t \le 1. \end{cases}$$

We will define λ_n as:

$$\lambda_{2n}(t) = \begin{cases} (1 - \frac{1}{2^{2n-1}})t \ \textit{if} & 0 \le t \le \frac{1}{2}, \\ (1 + \frac{1}{2^{2n-1}})t - \frac{1}{2^{2n-1}} \ \textit{if} & \frac{1}{2} < t \le 1. \end{cases}$$

$$\lambda_{2n+1}(t) = \begin{cases} (1 + \frac{1}{2^{2n}})t \ \textit{if} & 0 \le t \le \frac{1}{2}, \\ (1 - \frac{1}{2^{2n}})t + \frac{1}{2^{2n-1}} \ \textit{if} & \frac{1}{2} < t \le 1. \end{cases}$$

It is easy to see that

$$\sup_{0 \le t \le 1} |x_n(\lambda_n(t)) - x(t)| \to 0,$$

where

$$x(t) = \begin{cases} 0 \ \textit{if} & 0 \le t < \frac{1}{2}, \\ 1 \ \textit{if} & \frac{1}{2} \le t \le 1. \end{cases}$$

Let the sequence of random variables

$$\nu_n = \begin{cases} 2n \ \textit{with probability} & \frac{1}{2}, \\ 2n+1 \ \textit{with probability} & \frac{1}{2}. \end{cases}$$

Then for arbitrary $\mu_n \in \Delta[0,1]$ it holds:

$$\sup_{0 \le t \le 1} |x_{\nu_n}(\mu_n(t)) - x(t)| = \frac{1}{2}\Bigg(\sup_{0 \le t \le 1} |x_{2n}(\mu_n(t)) - x(t)| + \qquad (2)$$

$$\left.\sup_{0\le t\le 1}|x_{2n+1}(\mu_n(t))-x(t)|\right).$$

We will show there is no such $\mu_n\in\Delta[0,1]$ that (2) tands to zero.

It is easy to see that $\mu_n(\frac{1}{2})$ can't be equal to $\frac{1}{2}$. Assume that $\mu_n(\frac{1}{2})<\frac{1}{2}$. Then $x_{2n+1}(\mu_n(\frac{1}{2}))=0$ and $x(\frac{1}{2})=1$, which makes convergence impossible. Similarly, if $\mu_n(\frac{1}{2})>\frac{1}{2}$ then $x_{2n}(\mu_n(\frac{1}{2}))=0$.

In the following we will consider some corollaries of main result.

Corollary 1 *Let X_i be the independent random variables such that $\sum_{i=1}^n X_i\xrightarrow{d} X$ and $\pi(t)$ is an independent of X_i ($i\in\mathbb{N}$) Poisson random process, $E\pi(t)=t$ and $a>0$. Then it holds*

$$\sum_{i=1}^{\pi(n(t+a))} X_i\xrightarrow{d} X \text{ as } n\to\infty \text{ in } D[0,1].$$

Proof. Note that $\frac{\pi(n(t+a))}{n}\xrightarrow{d} t+a$ in $D[0,1]$ as $n\to\infty$. Thus all conditions of Theorem 1 are satisfied.

Let X_i be the i.i.d. random variables such that $EX_i=0, DX_i=\sigma^2$ and

$$S_i=\sum_{k=1}^{i}X_k.$$

Consider the random processes

$$X_n(t)=\frac{1}{\sigma\sqrt{n}}S_{i-1}+\frac{t-(i-1)/n}{1/n}\frac{1}{\sigma\sqrt{n}}X_i,\ t\in\left[\frac{i-1}{n},\frac{i}{n}\right].$$

It is well-know that $X_n\xrightarrow{d} W$ as $n\to\infty$ in $D[0,1]$ (here by W denoted a Wiener random process).

Corollary 2 [*generalized invariance principle*] *Let ν_n is a sequence of random elements taking values in $\mathbb{N}$, $f(n)$ is a function taking values in $\mathbb{R}$ such that $f(n)\to\infty$ as $n\to\infty$ and*

$$\frac{\nu_n}{f(n)}\xrightarrow{d}\nu$$

and it exists $c>0$ such that $\nu>c$. Then it holds

$$X_{\nu_n}\xrightarrow{d} W \text{ as } n\to\infty \text{ in } D[0,1],$$

where

$$X_{\nu_n}(t)=\frac{1}{\sigma\sqrt{\nu_n}}S_{i-1}+\frac{t-(i-1)/\nu_n}{1/\nu_n}\frac{1}{\sigma\sqrt{\nu_n}}X_i,\ t\in\left[\frac{i-1}{\nu_n},\frac{i}{\nu_n}\right].$$

Remark 2 *Random process $X(t)$ is usually interpreted as a process of random walk of a particle, which changes direction at moments of times $t = \frac{i}{n}$. Random process X_{ν_n} can be interpreted as a random walk of a particle, which changes direction at random moments of time.*

3 Estimation of convergence rate

Theorem 2 *Let Y_n, Y and ν_n, ν are the random processes in $D[0,1]$, $\nu_n(t)$ has non-decreasing non-negative trajectories, for every $t \in [0,1]$ $\nu_n(t)$ take values in $\mathbb{N}$, $f(n)$ is a function taking values in $\mathbb{R}$ and besides it exists $c > 0$ such that $\nu(t) > c$ for all $t \in [0,1]$. Let Y_n and ν_n are independent. Then*

$$\sup_x |P\{Y_{\nu_n} < x\} - P\{Y < x\}| \le \sup_x \sup_{k \ge [cf(n)]} |P\{Y_k < x\} - P\{Y < x\}| +$$

$$+2 \sup_x \sup_k |P\{Y_k < x\} - P\{Y < x\}| \inf_x \left| P\{\frac{\nu_n}{f(n)} < x\} - P\{\nu < x\} \right|.$$

Proof. Let $x \in \mathbb{R}$ is arbitrary. Then

$$|P\{Y_{\nu_n} < x\} - P\{Y < x\}| = \left| \sum_{k=1}^{\infty} (P\{Y_k < x\} - P\{Y < x\}) P\{\nu_n = k\} \right| \le$$

$$\sum_{k=1}^{\infty} |P\{Y_k < x\} - P\{Y < x\}| P\{\nu_n = k\} \le$$

$$\sum_{k=1}^{\infty} |P\{Y_k < x\} - P\{Y < x\}| P\left\{ \frac{k-1}{f(n)} < \nu \le \frac{k}{f(n)} \right\} + \tag{3}$$

$$\left| \sum_{k=1}^{\infty} |P\{Y_k < x\} - P\{Y < x\}| \left(P\left\{ \frac{k-1}{f(n)} < \nu_n \le \frac{k}{f(n)} \right\} - P\left\{ \frac{k-1}{f(n)} < \nu \le \frac{k}{f(n)} \right\} \right) \right|.$$

Note that for all $k < cf(n)$ the equality $P\{\nu \le \frac{k}{f(n)}\} = 0$ is holds. Then the first term of (3) can be estimated by

$$\sum_{k=1}^{\infty} |P\{Y_k < x\} - P\{Y < x\}| P\left\{ \frac{k-1}{f(n)} < \nu \le \frac{k}{f(n)} \right\} \le \max_{k \ge cf(n)} |P\{Y_k < x\} - P\{Y < x\}|.$$

Let $N \in \mathbb{N}$ is arbitrary. For the second term of (3) we have

$$\left| \sum_{k=1}^{\infty} |P\{Y_k < x\} - P\{Y < x\}| \left(P\left\{ \frac{k-1}{f(n)} < \nu_n \le \frac{k}{f(n)} \right\} - P\left\{ \frac{k-1}{f(n)} < \nu \le \frac{k}{f(n)} \right\} \right) \right|$$

$$\leq \sup_k |P\{Y_k < x\} - P\{Y < x\}| \left| P\left\{\nu_n \leq \frac{N}{f(n)}\right\} - P\left\{\nu \leq \frac{N}{f(n)}\right\}\right| +$$

$$\sup_k |P\{Y_k < x\} - P\{Y < x\}| \left| P\left\{\nu_n > \frac{N}{f(n)}\right\} - P\left\{\nu > \frac{N}{f(n)}\right\}\right| =$$

$$= 2\sup_k |P\{Y_k < x\} - P\{Y < x\}| \left| P\left\{\nu_n \leq \frac{N}{f(n)}\right\} - P\left\{\nu \leq \frac{N}{f(n)}\right\}\right|.$$

The arbitrariness of N implies the following inequality for second term of (3)

$$\left|\sum_{k=1}^{\infty} |P\{Y_k < x\} - P\{Y < x\}| \left(P\left\{\frac{k-1}{f(n)} < \nu_n \leq \frac{k}{f(n)}\right\} - P\left\{\frac{k-1}{f(n)} < \nu \leq \frac{k}{f(n)}\right\}\right)\right|$$

$$\leq 2\sup_k |P\{Y_k < x\} - P\{Y < x\}| \inf_x \left| P\{\frac{\nu_n}{f(n)} < x\} - P\{\nu < x\}\right|.$$

That completes the proof.

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